SPATIAL EXTERNALITIES AND THE STABILITY OF INTERACTING POPULATIONS NEAR THE CENTER OF A LARGE AREA*

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1. INTRODUCTION

In this paper we analyze the spatial dynamics of a class of systems in which many agents continually reassess decisions of where to locate in a given region. Each agent gives rise to an externality whose impact diffuses to other agents. Thus every agent experiences a composite of externalities emitted by other agents. We term this composite a spatial externality. On the one hand, the level of the externality at any location depends on the spatial distribution of the population. On the other hand, the distribution of the externality induces changes in the distribution of the population. It proves of interest to analyze this joint unfolding of a population distribution and an externality distribution, particularly for the case of several classes of interacting agents.

It is well established that the manner in which urban activities adjust to spatial externalities represents a problem at the core of current urban research priorities. Here we extend previous analyses of spatial equilibrium and adjustment dynamics in systems influenced by such externalities [see, for example, Miyao (1978)]. The main object of our concern is the relation between the structure of the externality diffusion operator and the form and stability of the equilibrium population distribution. The externality diffusion operator summarizes the processes by which an externality diffuses away from some agent in the space. Hence, the operator possesses a spatial structure. The stability of an equilibrium population distribution refers to a situation in which small perturbations to the distribution disappear over time. An implicit belief underlying our use of stability analysis is that unstable equilibrium distributions tend not to persist for any significant length of time; hence, conditions that lead to stability have an inherent interest. In particular, it is of interest to relate the conditions for stable population distributions and the spatial structure of the externality diffusion operator.

Our plan is the following. We first display the model in full generality and briefly discuss the existence and uniqueness of the spatial equilibria. At this level of generality, however, it proves difficult to discuss necessary and sufficient conditions for the stability of the spatial equilibria, which is the main object of our

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concern. In consequence, we focus our attention on the case of two classes of interacting populations near the center of a large area. Under these circumstances we first examine the nature of spatial equilibria and derive necessary and sufficient conditions for the stability of a system with spatially uniform income distributions. We then relax this assumption by introducing small but spatially arbitrary variations in income and again examine the spatial structure of the equilibrium population distributions as well as the associated criteria for stability.

The necessary and sufficient conditions for stability derived in our study provide some general insights into relations between the structure of spatial interaction and the stability of population distributions. For example, under the conditions of our analysis, the stability of the population distributions does not depend on the spatial structure of the externality diffusion operator, although it does depend on the aggregate effect of the operator. Furthermore, the stability criteria take essentially the same form for one- or two-dimensional distributions of population. Finally, while small redistributions of income in a system with an initially uniform distribution of income have no effect on the criteria for stability, small changes in the aggregate income of the system may be sufficient to cause destabilization of the population distribution.

2. THE MODEL

We begin with a set of simple conservation conditions, one for each of \( K \) classes of agents at some location. Agents are assumed to be distributed over \( L \) distinct locations. These conditions express the expected behavior of a system in continuous time and may be written

\[
\frac{dn_k^l}{dt} = - \sum_{m=1}^{L} n_k^l p_{lm} + \sum_{n=1}^{L} n_m^k p_{mn}^k
\]

where \( n_k^l \) is the number of individuals of class \( k \) at location \( l \), and \( p_{lm} \) is the probability that an individual of class \( k \) will move from location \( l \) to location \( m \) in a unit interval of time.

We assume no migration across the boundaries of the region, hence

\[
\sum_{m=1}^{L} p_{lm} = 1
\]

which we use to rewrite (1) as

\[
\frac{dn_k^l}{dt} = - n_k^l + \sum_{m=1}^{L} n_m^k p_{ml}^k
\]

Since births and deaths are ignored in (1), Equations (2) and (3) imply that the total number of individuals in each class \( k \)

\[
\sum_{l=1}^{L} n_k^l = N_k
\]

is constant over time.

The relocation probabilities \( p_{lm} \) generally are assumed to depend on both the
utilities of relocation for each class of agent $u_{lm}^k$ and on the spatial distribution of the $K$ classes of agents

$$p_{lm}^k = p^k[u_{l}^k, n]$$

where $u_{lm}^k \in u_i^k$ and $n_l^k \in n$. The $K$ sets of (indirect) utility functions may be interpreted as representative utilities for each class of agent, representing an assessment of the utility that an agent would attain on moving from location $l$ to location $m$. These utility functions are assumed to depend on disposable income at the destination after relocation costs, $y_{lm}^k$; on the cost of shelter at the destination $r_{m}^k$; and on the level of the externality at the destination $E_{m}^k$. In particular

$$u_{lm}^k = u^k[y_{lm}^k - r_{m}^k, E_{m}^k]$$

It also is assumed that the shelter costs depend on the number of individuals of different classes at location $m$ while the externality depends on the spatial distribution $n$.

The introduction of the utility functions into (5) needs no justification. Relocation probabilities, however, may be expected to depend on information flows concerning welfare opportunities. For example, information relating to job opportunities may be a prerequisite to many relocations. Assuming that the information flows to a given location depend on the spatial distribution of agents, we introduce the distribution $n$ as an argument of the relocation probabilities (5).

We now specify certain parts of the general model (3)-(6) in greater detail. In particular, we specify a functional form for the probabilities (5), the arguments of the utility functions (6), and the geometry of the regional in which the population exists.

In order to specify a functional form for the probabilities, we impose the following requirement: that in the absence of relocation costs, any steady state distribution of the population, as determined by

$$\frac{dn_l^k}{dt} = 0$$

also be a spatial equilibrium in the sense that

$$u_{lm}^k = u_{m}^k$$

$$= \bar{u}^k$$

where (8) follows from the absence of relocation costs.\footnote{The distinction between the concepts of steady state and spatial equilibrium is the following. From (1) and (7), a steady state is characterized by a global balance between corresponding aggregate inflows and outflows. A spatial equilibrium is characterized by a global balance between corresponding marginal costs and marginal benefits arising from changes in location. Since, for every class of agent, such global balance implies that welfare is maximized everywhere, a spatial equilibrium of the system is characterized by a global equality of welfare within classes as suggested by $\bar{u}^k$. Clearly a spatial equilibrium is a steady state, but a steady state may not be a spatial equilibrium.}

The requirements (8) and (9) impose restrictions on the form of the probabili-
ties. A form consistent with these requirements is

\[ P_m^k = \frac{n_m^k u_m^k}{\sum_i n_i^k u_i^k} \]

(10)

It is easily verified using the probabilities (10) that any steady state of (3) is also a spatial equilibrium. We write the externality in the form

\[ E_m^k = \sum_{j=1}^K \sum_{i=1}^L f_{ji} n_i^j \]

(11)

in which \( f \) is a distance response function for the externality. The term \( f_{ji} n_i^j \) represents the impact of the externality generated by individuals of the \( j \)th class located at place \( l \) on some individual of the \( k \)th class located at place \( m \). Thus every agent experiences a composite of externalities emitted by other agents, and every agent emits an externality whose impact diffuses to other agents.

Finally, we assume that our system is distributed on the plane, partitioned by a regular square grid into cells or locations. For analytical convenience the distance response function is assumed to depend only on distance between cells

\[ f_{im}^{jk} = f_{||im||} \]

(12)

where \( ||l, m|| \) is some metric defined on the cells. A general representation for such a distance-response function is shown in Figure 1. In this figure, for arbitrary \( l \), locations have been partitioned according to distance from \( l \) and labelled \( f_{im}^{jk} = f_{1m}^{jk} \); \( f_{im}^{jk} = f_{2m}^{jk} \) for location cells that are nearest neighbors of \( l \); \( f_{im}^{jk} = f_{3m}^{jk} \) for the next nearest neighbors of \( l \); and so on. The result is a distance-response function. The spatial domain of the function is determined by the pattern of nonempty cells. An empty cell, corresponding to \( f_{im}^{jk} = 0 \), indicates that there is no influence of class \( k \) on class \( j \) at the given distance. Different classes will generate different distance-response functions about an agent of class \( k \).

The form of the distance-response function for an agent of a given class is

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assumed to be invariant under translation. We also assume that with a sufficiently great distance of separation between location \( l \) and \( m \)

\[
f_{||l,m||}^{jk} \to 0
\]

and that the total effect of the distance-response function is finite in the sense that

\[
\left| \sum_m f_{||l,m||}^{jk} \right| = \left| \sum_l f_{||l,m||}^{jk} \right| = |F^{jk}| < \infty
\]

In order to circumvent the analytical problems arising from boundary conditions at the edges of our region, we investigate the behavior of the specific model close to the center of a large region. Such conditions are approximated by placing the system on a torus,\(^4\) and by taking \( l \) sufficiently large that the distance-response function approaches zero after some finite distance

\[
d_0 = \| l, m \|
\]

that is small with respect to the size of the torus, or

\[
d_0 / \sqrt{L} \ll 1
\]

By extending the boundaries of the region in an indefinite sense, we effectively diffuse the effects of the boundary conditions over larger and larger areas. In the

\(^4\)This is equivalent to folding a square plane into a tube and joining the ends of the tube.
limit, the effects of boundary conditions will be negligible near the center of the region. This effect is due to the relative localness of the distance response function.

3. SOME PROBLEMS AND STRATEGIES

It is possible to discuss the existence and uniqueness of the equilibrium population distributions at the level of generality represented by the model (3)-(6). We report the results of such an analysis in the Appendix. At this level of generality, however, it is difficult to discuss necessary and sufficient conditions for stability. In consequence, we focus attention on the stability of the spatial equilibria of two classes of interacting agents. We assume the behavior of the system to be governed by the specific model.

\[
\frac{dn_i^k}{dt} = -n_i^k + N^k \frac{n_i^k u_i^k}{\sum_j n_j^k u_j^k}
\]

(17)

\[
u_i^k - u_i^k \left[ y_i^k - r^k[n_1] \right], \quad \sum_i \sum_m f_{im} n_m^i
\]

(18)

where \( n_i^k \) is \( n_i \). We also assume that there are no costs of relocation, and that both classes of agents are located close to the center of a large area.

A problem of particular interest concerns the relationship between the form of the distance response function and the form and stability of the equilibrium population distributions. We approach the problem of stability by asking under what conditions small, but spatially arbitrary, perturbations to the spatial equilibria either grow or decay. It proves expositionally convenient to adopt the following two-stage strategy in discussing stability. In the first stage, we consider a system with spatially uniform incomes

\[
y_i^k = y_0^k
\]

(19)

We may interpret this condition to imply a system without transport costs. In the second stage, we introduce small, but spatially arbitrary, variations in income

\[
y_i^k = y_0^k + \epsilon_i^k
\]

(20)

where \( \epsilon_i^k \) may be interpreted as the cost of transport, incurred in living at a given location. As a final point of strategy, it is convenient to prove results for the case of one spatial dimension, and extend the results to a system of two spatial dimensions.

4. SPATIAL EQUILIBRIUM AND STABILITY IN SYSTEMS WITH TWO CLASSES OF AGENTS: RESULTS

In this section we state the results of our analysis. In the next section, we outline the proofs for our assertions.

4.1 Results for a Spatially Uniform Distribution of Income

Under the conditions of a spatially uniform income (19), it is always possible to characterize explicitly one spatial equilibrium:
Proposition 1: For the system (17)–(18)

\[ \overline{n}_k^i = \overline{n}_0^k = N^k / L \]

is a spatial equilibrium. If sufficient conditions for uniqueness of the spatial equilibrium hold, then the only spatial equilibrium is a uniform population distribution for both classes of agents. Necessary and sufficient conditions for the stability of a uniform spatial equilibrium are given in

Proposition 2: For two classes of agents in a system described by (17)–(18), with relocation probabilities (10) differentiable in \( n \), the uniform spatial distribution (21) is stable with respect to small, but spatially arbitrary perturbations if and only if \( Re \lambda \) is negative, where

\[ \lambda_0 = \phi^{11} + \phi^{22} + \sqrt{(\phi^{11} + \phi^{22})^2 + 4 \phi^{12} \phi^{21}} \]

\[ \phi^{ij} = \frac{\overline{n}_0^j}{\overline{u}_0^i} \left( \frac{\partial u^i}{\partial D} \left( -\frac{\partial r^i}{\partial n^i} \right) + \frac{\partial u^i}{\partial E} F^{ij} \right) \]

and where \( D \) represents disposable income (net of shelter costs).

There are several points of interest to note in connection with these results. First, the linearized solutions exhibit oscillatory behavior in time if and only if

\[ (\phi^{11} + \phi^{22})^2 + 4 \phi^{12} \phi^{21} < 0 \]

Second, these stability criteria hold whether the system is distributed on a circle or on a torus. A third and major point of interest is that the stability criteria are independent of the spatial form of the distance-response functions and depend only on the aggregate effect

\[ F^{jk} = \sum_{l \neq m} f_{lm}^{jk} \]

Hence, distinct distance response functions give rise to the same stability criteria as long as they are equivalent in terms of the aggregate effect \( F^{jk} \).

The results are easier to interpret when specialized to one class of agent, in which case the uniform spatial equilibrium is stable if and only if

\[ \frac{\partial u}{\partial D} \left( -\frac{dr}{dn} \right) + \frac{\partial u}{\partial E} F < 0 \]

Suppose that \( \partial u/\partial D > 0 \) and \( dr/dn > 0 \). Then the system is certainly stable in the case of a negative externality. In the case of a positive externality, however, the system may become unstable if, for example, the externality experienced at each point is sufficiently large. In this case agglomeration will occur. Given the form of the distance response function, the spatial range of the externality must be sufficiently large to ensure agglomeration. Strongly localized externalities, on the other hand, increase the likelihood of stability because they correspond to relatively small \( F \). We also note that the stability criterion may change sign as the
number \( N \) of individuals in the system changes, since the partial derivatives depend on \( N/L \).

It is of interest to relate the stability criteria (22), (23) with the sufficient conditions for the uniqueness of the spatial equilibrium (77) developed in the Appendix. It is straightforward to show that the spatial equilibrium \( N/L \) is unique if

\[
\frac{\tilde{n}_0}{u_0} \left[ \frac{\partial u}{\partial D} \left( \frac{dr}{dn} \right) + \frac{\partial u}{\partial E} F \right] \leq \frac{\tilde{n}_0}{u_0} \frac{\partial u}{\partial E} (F - f_0)
\]

It should be noted that the LHS of (27) is the criterion for stability, while the first term on the RHS represents the impact of the externality from all areas except the own area. Hence, as long as the effect of the externality diffusing from the outside areas is not too large, then the uniqueness of the uniform equilibrium distribution implies the stability of the equilibrium distribution. A final point of interest concerning the results for one class of agent is that oscillatory behavior in time is no longer possible for the perturbed solutions.

4.2 Results for a Nonuniform Distribution of Income

The results of the previous section provide a basis for results concerning a system with small but spatially arbitrary deviations from a uniform income distribution.

It is always possible to characterize in an explicit manner the following nonuniform steady state.

**Proposition 3:** If the uniform income distribution \( \phi_0 \) is perturbed by small, but arbitrary amounts \( \epsilon_i \), then an approximate steady state solution to the system (17)--(18) is given by

\[
\bar{n}_i^k = \bar{n}_0^k + \frac{n_0^k}{u_0} \phi_{ij} \frac{\partial u^k}{\partial D} (\epsilon_{aw} - \epsilon_i) - \frac{n_0^k}{u_0} \phi_{ij} \frac{\partial u^j}{\partial D} (\epsilon_{aw} - \epsilon_i)
\]

where

\[
\Phi = \phi^{11} \phi^{22} - \phi^{12} \phi^{21}
\]

is assumed to be nonzero and where \( \epsilon_{aw} \) is the average value of the perturbation to income over the region.\(^5\)

These results are easily specialized to the case of one class of agent

\[
\bar{n}_i = \bar{n}_0 + \frac{\partial u}{\partial D} \left( \frac{dr}{dn} + \frac{\partial u}{\partial E} F \right) (\epsilon_{aw} - \epsilon_i)
\]

Hence, in the case of one class of agent, the perturbation to the uniform

\(^5\)The steady state of Proposition 3 is approximate in the sense that the errors of the approximation are \( O(\epsilon^2) \).
distribution of population at location \( l \) is directly proportional to the deviation of the perturbed income at that location from the average value of the perturbed income in the whole system. While a similar interpretation holds when two classes of agents are involved, spatial variations in the numbers of individuals of a given class at location \( l \) depend on the income perturbations affecting both classes of agents at that location. Furthermore, income perturbations at other locations do not affect population perturbations at location \( l \), while the externality distance-response functions affect the population distributions only by way of the aggregate response \( F \).

Necessary and sufficient conditions for the stability of these nonuniform steady states with respect to small but arbitrary perturbations in the population distribution are given in

**Proposition 4:** For two classes of agents in a system described by (17)-(18), with relocation probabilities (10) differentiable in \( n \), the steady state (28) is stable with respect to small, but spatially arbitrary perturbations if and only if \( \text{Re} \lambda_1 \) and \( \text{Re} \lambda_2 \) are negative, where

\[
\lambda_1 - \lambda_0 + \frac{(1 + \bar{\lambda}_1)(\xi_{11} + \bar{\xi}_{12} + \bar{\xi}_{21} + \bar{\xi}_{22}) - (\bar{\xi}_1 + \bar{\xi}_2)(\xi_{21} + \bar{\xi}_2\bar{\xi}_{12} + \bar{\xi}_1\bar{\xi}_2 + \xi_{22})}{(1 + \bar{\lambda}_1)(1 + \bar{\lambda}_2) - (\bar{\xi}_1 + \bar{\xi}_2)^2} \\
\lambda_2 = \lambda_0 + \frac{(1 + \bar{\lambda}_2)(\bar{\xi}_{11} + \bar{\xi}_{12} + \bar{\xi}_{21} + \bar{\xi}_{22}) - (\bar{\xi}_1 + \bar{\xi}_2)(\bar{\xi}_{21} + \bar{\xi}_2\bar{\xi}_{12} + \bar{\xi}_1\bar{\xi}_2 + \xi_{22})}{(1 + \bar{\lambda}_1)(1 + \bar{\lambda}_2) - (\bar{\xi}_1 + \bar{\xi}_2)^2} \\
\xi_1 = \frac{\left(\frac{\bar{u}_1^2}{n_0} - \frac{\bar{u}_0^2}{n_0}\phi^{11}\right) + \sqrt{\left(\frac{\bar{u}_1^2}{n_0} - \frac{\bar{u}_0^2}{n_0}\phi^{11}\right)^2 + 4\left(\frac{\bar{u}_1^2}{n_0} - \frac{\bar{u}_0^2}{n_0}\phi^{12}\right)}}{2\left(\frac{\bar{u}_0^2}{n_0}\phi^{12}\right)} \\
\xi_2 = \frac{\left(\frac{\bar{u}_1^2}{n_0} - \frac{\bar{u}_0^2}{n_0}\phi^{22}\right) + \sqrt{\left(\frac{\bar{u}_1^2}{n_0} - \frac{\bar{u}_0^2}{n_0}\phi^{22}\right)^2 + 4\left(\frac{\bar{u}_1^2}{n_0} - \frac{\bar{u}_0^2}{n_0}\phi^{12}\right)}}{2\left(\frac{\bar{u}_0^2}{n_0}\phi^{21}\right)} \\
\xi_{ij} = \left(\sum_i \xi_i\right) \frac{\bar{n}_0}{\bar{u}_0} \left(\frac{\partial u_i}{\partial D} - \frac{\partial u_i}{\partial n}\right) \left(\frac{\partial^2 u_i}{\partial D^2} - \frac{\partial u_i}{\partial D}\right) \left(\frac{\partial u_i}{\partial E}\right) \left(\frac{\partial^2 u_i}{\partial D\partial E} - \frac{\partial u_i}{\partial D}\right) \left(\frac{\partial u_i}{\partial E}\right)
\]

where \( \lambda_0 \) is the stability criterion of the uniform steady state (22), and it is assumed that \( \xi_1\xi_2 \neq 1 \). In these relations all derivatives are evaluated at the uniform steady state.

While these relations are evidently complex, they are characterized by several
points of some interest. First, the same results hold irrespective of whether the system is distributed on a circle or on a torus. Second, the distance-response functions enter the relations (31)–(35) only by way of their aggregate values $F$. Hence, the results are independent of the spatial structure of the externality diffusion operator and depend only on the aggregate effects of the externality at any location.

A third point of interest concerns the manner in which the aggregate perturbations to income in the system, $\Sigma_i \epsilon_i$, affect the stability criteria (31)–(32). It may be noted that if

$$\sum_i \epsilon_i = 0$$

then the stability criterion for the nonuniform spatial equilibrium is exactly the same as the criterion for the uniform spatial equilibrium. In other words, a (small) redistribution of income does not affect the stability of the spatial equilibrium. On the other hand, (small) additions or subtractions from the aggregate income in the system modify the stability characteristics of the uniform steady state.

5. PROOFS OF THE RESULTS

We outline proofs for the propositions stated above. The stability results are obtained by linearizing Equations (17)–(18) about the appropriate spatial equilibria, and by finding the eigenvalues of the resulting coefficient matrices. It is convenient to prove the stability results first for the case of a system distributed on a circle. The extension of these results to a system distributed on a torus follows easily.

5.1. The Case of a Spatially Uniform Distribution of Income

By substitution, it is straightforward to show that the uniform population distribution (21) leads to a spatially uniform level of utility when $y^k_i = y^k$ for all $l$, so producing a spatial equilibrium by (17).

On linearizing (17) about the uniform spatial equilibrium, one obtains

$$\frac{d\hat{n}_i^k}{dt} = \frac{\hat{n}_0^k}{U_0} \left( - \frac{\partial r^k}{\partial n^k} \right) \left( \hat{n}_i^k - \frac{\sum_m \hat{n}_m^k}{L} \right) + \frac{\partial u^k}{\partial E} \sum_m \left( f_{lm}^{jk} - \frac{F^{kk}}{L} \right) \hat{n}_m^j + \frac{\partial u^k}{\partial D} \left( - \frac{\partial r^k}{\partial n^k} \right) \left( \hat{n}_i^k - \frac{\sum_m \hat{n}_m^k}{L} \right) + \frac{\partial u^k}{\partial E} \sum_m \left( f_{lm}^{jk} - \frac{F^{kk}}{L} \right) \hat{n}_m^j \quad (j \neq k)$$

where $\hat{n}_i^k$ is the perturbation to the uniform population distribution. In matrix form, we have

$$\begin{bmatrix} \frac{d\hat{n}^1}{dt} \\ \frac{d\hat{n}^2}{dt} \end{bmatrix} = \begin{bmatrix} C^1 & C^2 \\ C^3 & C^4 \end{bmatrix} \begin{bmatrix} \hat{n}_1^1 \\ \hat{n}_2^2 \end{bmatrix}$$
where each $C^i$ is an $l \times l$ circulant matrix [see Davis (1979)], since the distance response functions $f_{lm}^{ih}$ depend only on distance, the remaining coefficients are independent of location and the system is on a circle.

We find the eigenvalues of the matrix of coefficients in (38) by first diagonalizing each submatrix $C^i$ with a similarity transformation. We simplify the resulting matrix using the condition that our system be near the center of a large plane region, and force the resulting matrix into lower triangular form using a second similarity transformation.

The transformation

$$
\begin{bmatrix}
A^1 & A^2 \\
A^3 & A^4
\end{bmatrix} = 
\begin{bmatrix}
V^+ & 0 \\
0 & V^+
\end{bmatrix}
\begin{bmatrix}
C^1 & C^2 \\
C^3 & C^4
\end{bmatrix}
\begin{bmatrix}
V & 0 \\
0 & V
\end{bmatrix}
$$

where $V$ is the well-known unitary matrix that diagonalizes a circulant, contains diagonal submatrices $A^i$ whose nonzero entries are given by

$$
\lambda_k^j = \sum_i c_i^j e^{2\pi i (i-1)k/L} \quad (j = 1, \ldots, 4)
$$

where $c_i^j$ is the first row of $C^j$.

Using the fact that each circulant $C^i$ is symmetrical (the distance response functions depend only on distance) and using the assumption that $f_{lm}^{ih}$ is zero for $\|l, m\| > d_0/\sqrt{L} \ll 1$, it is straightforward to show that

$$
\begin{align*}
\lambda_1^j &= 0 \quad (j = 1, \ldots, 4) \\
\lambda_2^1 &= \frac{\partial u^1}{\partial D} - \frac{\partial r^1}{\partial n} + \frac{\partial u^1}{\partial E} F_{11} = a \\
\lambda_2^2 &= \frac{\partial u^1}{\partial D} - \frac{\partial r^1}{\partial n} + \frac{\partial u^1}{\partial E} F_{21} = b \\
\lambda_3^1 &= \frac{\partial u^2}{\partial D} - \frac{\partial r^2}{\partial n} + \frac{\partial u^2}{\partial E} F_{12} = c \\
\lambda_3^2 &= \frac{\partial u^2}{\partial D} - \frac{\partial r^2}{\partial n} + \frac{\partial u^2}{\partial E} F_{22} = d
\end{align*}
$$

for $l = 2, L$.

Using the second transformation

$$
\begin{bmatrix}
I & X \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A^1 & A^2 \\
A^3 & A^4
\end{bmatrix}
\begin{bmatrix}
I & X^{-1} \\
0 & I
\end{bmatrix}
$$

where $X$ is an $L \times L$ diagonal matrix satisfying

$$
XXA^3 - X(A^1 - A^4) - A^2 = 0
$$

---

*See Bellman (1970).*
we force (39) into lower triangular form. The diagonal elements of this matrix are
the eigenvalues we seek
\[
\lambda_1 = \lambda_{L+1} = 0
\]
\[
\lambda_j = \frac{(a + d) + \sqrt{(a - d)^2 + 4bc}}{2} \quad (1 < j \leq L)
\]
\[
\lambda_j = \frac{(a + d) - \sqrt{(a - d)^2 + 4bc}}{2} \quad (L + 1 < j \leq 2L)
\]
Since the real part of (49) is at least as great as the real part of (50), Equation (49) provides the criterion for stability.

The extension of these results from the circle to the torus is easily accomplished by labelling areas in terms of a system of orthogonal circular coordinates on the torus. We obtain a set of linearized equations similar to (38), except that each \( C^i \) is now a circulant of circulant matrices. We apply three similarity transformations in order to obtain the eigenvalues of the coefficient matrix.

The first transformation diagonalizes each circulant within each \( C^i \), the second diagonalizes each \( C^i \), while the third reduces the resulting matrix to lower triangular form, exactly as in the case of a system on the circle.

5.2 The Case of a Nonuniform Distribution of Income

The conditions governing the spatial equilibria (7)-(9) for the case of perturbed (or nonuniform) incomes may be written explicitly as
\[
u^1[y^1 + \epsilon_1 - r^1[\bar{n}_0^1 + \bar{n}_1^1, \bar{n}_0^2 + \bar{n}_1^2]],
\]
\[
\sum_m f^{11}_{lm} (\bar{n}_0^1 + \bar{n}_1^1) + f^{21}_{lm} (\bar{n}_0^2 + \bar{n}_1^2)] = \bar{u}_0^1 + \bar{u}^1
\]
\[
u^2[y^2 + \epsilon_2 - r^2[\bar{n}_0^1 + \bar{n}_1^1, \bar{n}_0^2 + \bar{n}_1^2]], \sum_m f^{12}_{lm} (\bar{n}_0^1 + \bar{n}_1^1)
\]
\[
+ f^{22}_{lm} (\bar{n}_0^2 + \bar{n}_1^2)] = \bar{u}_0^2 + \bar{u}^2
\]
\[
\sum_i (\bar{n}_0^1 + \bar{n}_1^1) = N^1
\]
\[
\sum_i (\bar{n}_0^2 + \bar{n}_1^2) = N^2
\]
where \( \bar{n}_i^k \) is the perturbed part of the steady-state population distribution and \( \bar{u}_i^k \) is the perturbation to the utility level. On expanding these equations to first order in the \( \bar{n}_i^k \), we obtain a system of linear equations which take the form
\[
\begin{bmatrix}
C^1 & C^2 \\
C^3 & C^4
\end{bmatrix}
\begin{bmatrix}
\bar{n}^1 \\
\bar{n}^2
\end{bmatrix}
= 
\begin{bmatrix}
\epsilon^1 \\
\epsilon^2
\end{bmatrix}
\]

\[\text{We would like to express thanks to Marvin Marcus of the Institute for Interdisciplinary Applications of Algebra and Combinatorics, UCSB, for indicating this transformation.}\]
and

\begin{equation}
\sum_i \bar{h}_i = 0
\end{equation}

where each \( C^i \) is a circulant matrix.

Using the similarity transformation (39) and the assumption that the system lies near the center of a large plane region, we obtain

\begin{equation}
\begin{bmatrix}
aI & bI \\
cI & dI
\end{bmatrix}
\begin{bmatrix} V^+ & 0 \\ 0 & V^+
\end{bmatrix}
\begin{bmatrix} \bar{h}_1 \\ \bar{h}_2
\end{bmatrix} = \begin{bmatrix} \bar{V}^+ & 0 \\ 0 & \bar{V}^+
\end{bmatrix}
\begin{bmatrix} \epsilon^1 \\ \epsilon^2
\end{bmatrix}
\end{equation}

where \( a, b, c, d \) are defined in (42)-(45). Assuming that the inverse of the first matrix on the LHS exists, it is easy to solve (57) for \( \bar{h} \). Using the conditions (56), one may then solve explicitly for \( \bar{u}^k \), and so obtain the results of Proposition 3. The extension of the results from the circle to the torus is analogous to the extension of the previous stability results from the circle to the torus.

On linearizing (17) about the spatial equilibrium implicit in (57), one obtains, for the first class of agent

\begin{equation}
\begin{align*}
\frac{d\bar{u}_i}{dt} &= \frac{\bar{u}_i}{u_i} \left( \frac{\partial u_l}{\partial D} \left( - \frac{\partial r^l}{\partial n^l} \bar{h}_i \right) - \frac{1}{N^l} \sum_m \bar{n}_m \frac{\partial u_m}{\partial D} \left( - \frac{\partial r^m}{\partial n^m} \right) \bar{h}_m \\
&\quad + \sum_m \left( \frac{\partial u_l}{\partial E} f_{lm}^1 \bar{n}_m^1 - \frac{1}{N^l} \bar{n}_m \frac{\partial u_m}{\partial E} \sum_j f_{mj}^1 \bar{h}_j \right) + \frac{\partial u_l}{\partial E} \left( - \frac{\partial r^l}{\partial n^l} \right) \bar{h}_i \right) \bar{h}_i^2 \\
&\quad - \frac{1}{N^2} \sum_m \bar{n}_m \frac{\partial u_m}{\partial D} \left( - \frac{\partial r^m}{\partial n^m} \right) \bar{h}_m^2 + \sum_m \left( \frac{\partial u_m}{\partial E} f_{lm}^1 \bar{n}_m^2 - \frac{1}{N^2} \bar{n}_m \frac{\partial u_m}{\partial E} \sum_j f_{mj}^2 \bar{h}_j \right) \bar{h}_i^2 \right)
\end{align*}
\end{equation}

where \( \bar{n}_m^k \) are the nonuniform equilibrium population distributions and \( \bar{h}_i^k \) are the perturbations to these distributions. An analogous equation holds for the second class of individuals. In matrix form, we have

\begin{equation}
\begin{bmatrix}
\frac{d\bar{h}_i^1}{dt} \\
\frac{d\bar{h}_i^2}{dt}
\end{bmatrix} = \begin{bmatrix} M^1 & M^2 \\
M^3 & M^4
\end{bmatrix}
\begin{bmatrix} \bar{h}_i^1 \\
\bar{h}_i^2
\end{bmatrix}
\end{equation}

Perturbation Theory\(^8\) indicates that the eigenvalues of the coefficient matrix in (59) may be written in the form

\begin{equation}
\lambda_l = \lambda_{ql} + \hat{\lambda}_l \quad (l = 1, \ldots, 2L)
\end{equation}

One proceeds by expanding the coefficients in (59) about the uniform spatial equilibria (21), so obtaining to first order

\[^8\text{See Morse and Feshbach (1953).}\]
The perturbed parts of the eigenvalues \( \tilde{\lambda}_i \) are the eigenvalues of \( \mathcal{P} \). They may be expressed in terms of the following equation\(^9\)

\[
(62) \quad \mathcal{P}\omega_{0i} + \sum_{m=1}^{2L} \alpha_{mi} \lambda_{0m} \omega_{0m} = \tilde{\lambda}_i \omega_{0i} + \sum_{m=1}^{2L} \alpha_{mi} \lambda_{0i} \omega_{0m}
\]

where \( \omega_{0m} \) are the eigenvectors of the matrix \( \mathcal{C} \), \( \lambda_{0m} \) are the corresponding eigenvalues, \( \tilde{\lambda}_i \) are the perturbations to these eigenvalues due to the matrix \( \mathcal{P} \), and the numbers \( \alpha_{mi} \) are the coefficients used to express the perturbed eigenvectors in terms of the eigenvectors \( \omega_{0m} \).

A set of eigenvectors \( \omega_{0i} \) for \( \mathcal{C} \) must satisfy the conditions

\[
(63) \quad \mathcal{A} V^i \omega_{0i} = \lambda_{0i} V^i \omega_{0i}
\]

where

\[
\mathcal{A} = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}
\]

is the matrix defined in (39). A set of independent eigenvectors for \( \mathcal{A} \) can be found if

\[
(64) \quad \left(1 - \frac{a - \lambda_{01}}{b} \right) \left(1 - \frac{d - \lambda_{02}}{c} \right) = 1 - \xi_1 \xi_2 \neq 0
\]
in which case

\[
(65) \quad \omega_1 = (1, 0, \ldots, 0 | 0, \ldots, 0)
\]

\[
\omega_j = (0, \ldots, 1, \ldots, 0 | 0, \ldots, \xi_1, \ldots, 0) \quad (1 < j \leq M + 1)
\]

\[
(66) \quad \omega_{L+1} = (0, \ldots, 1, 0, \ldots, 0)
\]

\[
\omega_j = (0, \ldots, \xi_2, \ldots, 0 | 0, \ldots, 1, \ldots, 0) \quad (L + 1 < j \leq 2L)
\]

are a set of independent eigenvectors of \( \mathcal{A} \).

From (63), the eigenvectors \( \omega_{0i} \) of \( \mathcal{C} \) can be expressed in terms of the eigenvectors (65), (66) of \( \mathcal{A} \). While the eigenvectors \( \omega_{0i} \) are independent, they are not orthogonal, and in particular

\(^9\)See footnote 8.
while \( \omega_{0l} \) and \( \omega_{0l+1} \) are orthogonal to all. Hence, in order to isolate the perturbed eigenvalues \( \lambda_i \) in (62), it is necessary to take the inner product of both sides of (62) with \( \omega_{0l} \), \( \omega_{0l+1} \) and the inner product of

\[
(1 + \zeta_1) \omega_{0l} = 0 \quad (j \neq k; k \neq j + L)
\]

while \( \omega_{0l} \), \( \omega_{0l+1} \) are orthogonal to all. Hence, in order to isolate the perturbed eigenvalues \( \lambda_i \) in (62), it is necessary to take the inner product of both sides of (62) with \( \omega_{0l} \), \( \omega_{0l+1} \) and the inner product of

\[
(1 + \zeta_1) \omega_{0l} = 0 \quad (j \neq k; k \neq j + L)
\]

with \( \omega_{0l} \) and \( \omega_{0l+1} \) in order to eliminate the unknowns \( \alpha_{ml} \). On carrying out these required calculations and using (69)-(70) one obtains

\[
\lambda_i = \frac{(1 + \zeta_2) \omega_{0l} \mathcal{P} \omega_{0l} - (\zeta_1 + \zeta_2) \omega_{0l+1} \mathcal{P} \omega_{0l}}{(1 + \zeta_1)(1 + \zeta_2) - (\zeta_1 + \zeta_2)^2} \quad (1 \leq l \leq L)
\]

It remains now to compute the values of the bilinear forms in (72)-(73) using the facts that each submatrix \( \mathcal{P}^j \) of \( \mathcal{P} \) has a similar form, while a typical element \( p_{ij}^0 \) has a form in which many of the entries depend on none or one index of the matrix, or two indexes but on separate terms, or on terms of the form \( f^0_{ij} (\epsilon_{aw} - \epsilon_j) \). Since the eigenvectors \( \omega_{0l} \) take the form

\[
(l, \rho_j, \rho_j^2, \ldots, \rho_j^{L-1}, \zeta_1, \zeta_1 \rho_j, \ldots, \zeta_1 \rho_j^{L-1})
\]

where \( \rho_j \) is the \( j \)th of the \( L \) roots of unity, such terms will disappear from the quadratic forms, and we are left only with the terms expressed in Proposition 4.

The extension of the results to two dimensions is entirely analogous to the method described in the proof of Proposition 2.

REFERENCES


APPENDIX

In the Appendix we examine the issue of existence and uniqueness of steady state population distributions at the level of generality described by the model
It is straightforward to specialize these results for corresponding spatial equilibrium population distributions. Related arguments are extensions of Miyao (1978).

A. Existence

Proposition 5. If the probabilities (5) are continuous in \( n \) for \( 0 \leq n_i^k \leq N^n \), then there exists a steady state solution to the system (3)–(6).

From (4) and (7), a steady state is defined as

\[
\sum_m n_m^k p_{mi}^k = n_i^k
\]

Then

\[
\sum_l \sum_m n_m^k p_{mi}^k = \sum_m n_m^k \sum_l p_{mi}^k = \sum_m n_m^k = N^k
\]

Thus \( \Sigma_m n_m p_{mi} \leq N^k \) because \( n_i^k \geq 0 \) and \( p_{mi}^k \geq 0 \). In consequence, the function \( \Sigma_m n_m^k p_{mi}^k \) maps from a product of simplices with nonnegative elements to itself. According to the theorem of Brower, there is an \( n^* \) such that (75) holds. Then (76) ensures that \( n^* \) is indeed a steady state.

It should be noted that continuity is a crucial property and that one may construct models without steady states if this assumption is dropped.

B. Uniqueness

Proposition 6: If the probabilities (5) are differentiable in \( n \) for \( 0 \leq n_i^k \leq N^n \); if there are no two quantities \( (\partial/\partial n_i^j) \Sigma_m n_m^k p_{mi}^k \) with opposite sign for \( i \neq j \); and if

\[
\sum_k \frac{\partial}{\partial n_i^j} \sum_m n_m^k p_{mi}^k < \frac{1}{2} \quad \text{for} \quad \frac{\partial}{\partial n_i^j} \sum_m n_m^k p_{mi}^k < 0
\]

while

\[
\sum_{k \neq l} \frac{\partial}{\partial n_i^j} \sum_k n_m^k p_{mj}^k > -\frac{1}{2} \quad \text{for} \quad \frac{\partial}{\partial n_i^j} \sum_m n_m^k p_{mi}^k \geq 0
\]

then the steady state solution to the system (3)–(6) is unique.

Motivated by the definition of a steady state, we construct a mapping \( W: \Omega \rightarrow \mathbb{R}^{(k+1)l} \), where \( \Omega \) is the closed rectangular region \( \{ n \mid 0 \leq n_i^k \leq N^n \} \) and \( W \) is a vector of vectors with elements

\[
\omega_i^k = n_i^k - \sum_m n_m^k p_{mi}^k
\]

If the solution to \( W = 0 \) is unique, then the steady state is also unique.

By assumption \( W \) is differentiable in \( \Omega \). The Jacobian matrix of \( W, \partial [n] \), may be organized as a matrix of matrices, \( \partial [\omega_i^k] \); \( \partial [n_i^k] \), where

\[
J_{ij}^{kl} = \frac{\partial \omega_i^k}{\partial n_j^l}
\]
If

\[ J_{ij}^{ii} > \sum_{l \neq j} |J_{ij}^{ii}| + \sum_{k \neq i} \sum_{l} |J_{il}^{ik}| \text{ for all } i, j \]

then \( \delta \) has a dominant diagonal with positive diagonal elements for all \( n \) in \( \Omega \). It is known that \( W \) is univalent in \( \Omega \) (that is, if \( W[n] = W[n'] \) for \( n_1, n_2 \in \Omega \) then \( n_1 = n_2 \)). Then the solution to \( W = 0 \) must be unique. Using the previous definitions and taking into account that, because of (76)

\[ \frac{\partial}{\partial n_i^j} \left( \sum_{l} \sum_{m} n_m^i p_{ml}^k \right) = 0 \]

the condition for uniqueness of steady states (81) can be written as

\[ 1 - \frac{\partial}{\partial n_j^l} \sum_{m} n_m^i p_{mj}^i > \frac{\partial}{\partial n_j^l} \sum_{m} n_m^i p_{mj}^i + 2 \sum_{k \neq i} \frac{\partial}{\partial n_j^l} \sum_{m} n_m^k p_{mj}^k \]

when \( J_{ij}^{ii} > 0 \) for \( l \neq j \), or

\[ 1 - \frac{\partial}{\partial n_j^l} \sum_{m} n_m^i p_{mj}^i < \frac{\partial}{\partial n_j^l} \sum_{m} n_m^i p_{mj}^i - 2 \sum_{k \neq i} \frac{\partial}{\partial n_j^l} \sum_{m} n_m^k p_{mj}^k \]

when \( J_{ij}^{ii} < 0 \) for \( l \neq j \). These are equivalent to (77) and (79), respectively.

It should be noted that these conditions for uniqueness are only sufficient. It is easy to construct counter-examples that illustrate this point and for which a steady state exists although the conditions of Proposition 6 are not satisfied.

The quantity \( \partial (\Sigma_m n_m^i p_{mj}^k) / \partial n_j^l \) describes how the aggregate expected flow of agents of class \( k \) toward location \( l \) is affected by changes in the number of agents of class \( i \) in location \( j \). According to the first requirement of Proposition 6, any change in the number of agents of class \( i \) in location \( j \) should cause either increases or decreases in the aggregate expected flows of agents of class \( k \) toward any location other than \( j \). Unchanged flows are not ruled out, but any observed change in aggregate flow related to class \( k \) (other than those defined for location \( j \)) must be of the same sign. The only way changes in location \( j \) affect events in location \( l \neq j \) is through the consequent externality change (a composite of both information gathering and spatial externality changes) experienced at location \( l \). Hence, the first requirement limits the validity of Proposition 6 to cases where the combined externalities of class \( i \) on class \( k \) produce an effect either nonnegative or nonpositive throughout the landscape: effects that change from negative to positive with distance from the source (or vice-versa) are eliminated.

The two classes of conditions for uniqueness place definite constraints on the way spatial externalities diffuse. This stems from the series of components \( \partial p_{mj}^k / \partial n_j^l \) which may readily be expanded into

\[ \frac{\partial p_{mj}^k}{\partial n_j^l} = \frac{\partial p_{mj}^k}{\partial u_{mj}^l} \frac{\partial u_{mj}^l}{\partial n_j^l} + \sum_{i} \frac{\partial p_{mj}^k}{\partial u_{ml}^k} \frac{\partial u_{ml}^k}{\partial E_{ml}^k} f_{lj}^{ij} + \frac{\partial p_{mj}^k}{\partial n_j^l} \]

The absolute value of these components must be sufficiently small if uniqueness is to be ensured. Then, depending on the signs of the partial derivatives in (85), the
value of quantities $f_{ij}^s$ must be either sufficiently large or sufficiently small if uniqueness is to be ensured. In other words, depending on the case, the conditions for uniqueness impose either lower or upper bounds on the intensity of the spatial externality diffusion process.