Solutions of the bio-heat transfer equation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.


(http://iopscience.iop.org/0031-9155/33/7/002)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 130.113.111.210
The article was downloaded on 04/11/2010 at 17:57

Please note that terms and conditions apply.
Solutions of the bio-heat transfer equation

Wesley L Nyborg
Physics Department, Cook Physical Science Building, University of Vermont, Burlington,
VT 05405, USA

Received 3 November 1987, in final form 22 February 1988

Abstract. A solution of the bio-heat transfer equation for a 'step-function point source' is
presented and discussed. From this basic solution one can, in principle, obtain the
temperature field resulting from a general heat source distribution by superposition. As an
example, the method is used to calculate the temperature on the body surface at a point
where therapeutic ultrasound is applied. Comparison is made with experimental results
recently published by Williams and co-workers.

1. Introduction

In calculations of heat transport and temperature rise in perfused media much use is
made of a linear bio-heat transfer equation initially proposed by Pennes (1948). In
an assessment by Eberhart et al (1980) it is concluded that this equation is 'an adequate
model for prediction of the macroscopic temperature distribution in several biological
tissues'. It is convenient to write the bio-heat transfer equation as follows:
\[ \dot{T} = \kappa \nabla^2 T - T/\tau + q_v/c_v \]  \hspace{1cm} (1)
where \( q_v \) is the heat source function (the rate of heat production per unit volume), \( T \)
the temperature rise above the ambient level, \( \dot{T} \) the rate of temperature rise, \( \kappa \) the
thermal diffusivity, \( \tau \) the time constant for perfusion and \( c_v \) the volume specific heat
for tissue.

The thermal conductivity coefficient \( K \) differs from \( \kappa \) by the factor \( c_v \):
\[ K = c_v \kappa. \]  \hspace{1cm} (2)
The constant \( \tau \) is inversely proportional to the blood perfusion rate \( \omega \), a quantity in
units of mass divided by volume and time that has been cited in the literature (Roemer
et al 1984). Specifically, the perfusion time constant can be written as
\[ \tau = \rho_b c_v / \omega c_{vb} \]  \hspace{1cm} (3)
where \( \rho_b \) is the density of blood and \( c_{vb} \) is the volume specific heat for blood.

2. Point source solution

A useful solution of equation (1) was obtained by adapting a Green function, discussed
for a very different application by Morse and Feshbach (1953). This gives the tem-
perature rise at a distance \( r \) from a small source of volume \( dv \) that has been generating
heat at the rate \( q_v \), \( dv \) for a time \( t \), and can be written as
\[ T = (C/r)\{E[2 - \text{erfc}(\tau^* - R)] + E^{-1} \text{erfc}(\tau^* + R)\} \]  \hspace{1cm} (4)
where

\[ C = q_v \frac{dv}{8\pi K} \]  \hspace{1cm} (5) \\
\[ E = \exp(-r/L) \]  \hspace{1cm} (6) \\
\[ L = (\kappa \tau)^{1/2} \]  \hspace{1cm} (7) \\
\[ t^* = \frac{(t/\tau)^{1/2}}{} \]  \hspace{1cm} (8) \\
\[ R = \frac{r}{(4\kappa \tau)^{1/2}}. \]  \hspace{1cm} (9)

In equations (5)-(9) the quantities \( E, t^* \) and \( R \) are dimensionless, while \( C \) and \( L \) have the dimensions of length-temperature and length, respectively.

After a sufficiently long time \( t \), the function \( \text{erfc}[ \ ] \) in equation (4) reduces to zero, and one obtains for a steady state

\[ T = \frac{2CE}{r} = \left( \frac{2C}{r} \right) \exp(-r/L). \]  \hspace{1cm} (10)

The quantity \( L \) is a characteristic length for a perfusing medium and may be considered a 'perfusion length'. According to Swindell (1984) typical values of \( L \) (his \( \Lambda \)) vary from 20 mm for tissues (e.g. fat) in which the perfusion is moderate to 2 mm for tissues (e.g. brain) in which it is vigorous.

Using equation (10), another special result of interest can be obtained. Consider a large region uniformly filled with small heat sources which form a continuum in which \( q_v \) is constant. Determining the temperature rise at a point in the region by superposing contributions from surrounding sources, one arrives at the simple expression

\[ \hat{T} = q_v \frac{\tau}{c_v} \]  \hspace{1cm} (11)

for a steady state and uniform \( q_v \).

This result can also be obtained directly from equation (1) by considering \( \hat{T} \) and \( \nabla^2 T \) both equal to zero.

More information on the significance of \( \tau \) comes from equation (1) by considering a situation where heat sources have produced a temperature field that is nearly spatially uniform, after which all sources are simultaneously turned off, say, at \( t = 0 \). Because the temperature field is fairly uniform the thermal diffusion term \( \kappa \nabla^2 T \) in equation (1) is negligible. After sources are turned off, so that \( q_v \) is zero, equation (1) reduces to

\[ T = -\frac{T}{\tau} \]  \hspace{1cm} (12)

of which a solution is

\[ T = T_0 \exp(-t/\tau) \]  \hspace{1cm} (13)

where \( T_0 \) is the temperature rise at \( t = 0 \) when the sources are turned off. The quantity \( \tau \) can evidently be considered a 'perfusion time constant' for the medium.

If there is no perfusion, both \( \tau \) and \( L \) are infinite. Equation (4) then reduces to

\[ T = 2Cr^{-1} \text{erfc}(R). \]  \hspace{1cm} (14)

Plots of temperature elevation \( T \) against \( t \), based on equation (4), are shown in figures 1 and 2. In figure 1 the four curves give the calculated temperatures at different distances from the origin, all for a source generating 10 mW of heat and for \( \tau \) equal to 1000 s. At each distance \( r \) the temperature increases with time, approaching the
Solutions of the bio-heat transfer equation

1.2

-1.2

Temperature rise (°C)

-0.84 °C

-0.56 °C

-0.34 °C

0.434

Figure 1. Temperature plotted against time, according to equation (4), from a point source generating 10 mW of heat. The perfusion time constant \( \tau \) is 1000 s. Each curve is for a different distance \( r \): \( \cdots \cdots \), \( r = 1 \) mm; \( \cdots \cdots \cdots \), \( r = 1.4 \) mm; \( \cdots \cdots \cdots \cdots \), \( r = 2 \) mm; \( \cdots \cdots \cdots \cdots \cdots \cdots \), \( r = 3 \) mm. The number above and to the right of each curve gives the steady-state temperature, from equation (10), which is approached with increasing time.

Figure 2. Temperature plotted against time, as in figure 1, but at a distance \( r \) of 5 mm from a 100 mW source. Each curve is for a different value of the perfusion constant \( \tau \): \( \cdots \cdots \cdots \cdots \cdots \cdots \), \( \tau = 1000 \) s; \( \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \), \( \tau = 300 \) s; \( \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \), \( \tau = 100 \) s; \( \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots ..steady-state limit given by equation (10), shown here above and to the right of each curve. The time required to reach 50% of the limiting temperature rise increases with \( r \), being 6.5, 12, 22 and 42 s at distances \( r \) of 1, 1.4, 2 and 3 mm, respectively.

In figure 2 the distance \( r \) is 5 mm and the source output is 100 mW for all four curves, but the perfusion time constant \( \tau \) takes on different values. At small times all the curves nearly coincide; here the heat transport is governed primarily by diffusion. As time goes on, perfusion becomes more and more dominant. Comparing the curves, we see that for a highly perfused medium, i.e. one characterised by small \( \tau \), the final steady-state temperature is lower than for a less perfuse medium, but is reached more quickly. The time required to reach 50% of the limiting temperature rise increases with \( \tau \), being 36, 44, 68 and 90 s for values of \( \tau \) of 50, 100, 300 and 1000 s, respectively.
3. Superposition of point source solutions

Since equation (1) is a linear equation, solutions can be superposed. Hence when heat generation occurs over an extended region the temperature can be calculated at any point by adding contributions from all parts of the region using equation (4). In general, this requires an integration over three coordinates. However, the analysis is simplified when the \( q_v \) distribution possesses cylindrical symmetry, and the temperature calculations are made at points on the axis. Such symmetry exists when a circular focused or unfocused transducer with its axis along \( z \) radiates into a homogeneous absorbing medium. The temperature at any point on the axis can be obtained by summing contributions from infinitesimal rings of volume \( dv \) centred on the axis.

For clarity in discussing this procedure, the quantity \( z \) will indicate the position of an observation point, i.e. the distance from the transducer to a point on the axis at which the temperature is to be calculated; the quantity \( z^* \) will designate the distance along the axis from the transducer to a plane containing a heat source. Consider a specific ring within which heat is being generated, so that it acts as a heat source. Its radius is \( x \), while its infinitesimal thickness is \( dx \) in the \( x \) direction and \( dz^* \) in the \( z \) direction. It lies between planes perpendicular to the axis at \( z^* \) and \( z^* + dz^* \), as shown in figure 3. Its contribution to the temperature at \( z \) is given by equation (4), with \( T \) recognised as an infinitesimal temperature rise \(dT \) and \( r \) as the distance from any point on the ring to the axial point at \( z \). Thus

\[
r = \left[ x^2 + (z - z^*)^2 \right]^{1/2}
\]

(see figure 3). The volume of the ring, whose cross-sectional dimensions are \( dx \) and \( dz^* \), is \( 2\pi x \, dx \, dz^* \). To obtain the temperature at \( z \) an integration is carried out over the region of interest:

\[
T = \int \frac{q_v}{8\pi K r} \{ \} \, dv
\]
or

\[
T = \frac{1}{4K} \int \int \frac{q_v x}{r} \{ \} \, dx \, dz^*
\]

where \( \{ \} \) is the quantity in curly brackets on the right-hand side of equation (4).

![Figure 3. Defining sketch. An infinitesimal ring source of heat of radius \( x \) lies in a plane at a distance \( z^* \) from the transducer; the temperature in its field is calculated at a point on the axis whose distance from the transducer is \( z \), and whose distance from any point on the ring is \( r \).](image)
3.1. An example

For an application of the procedure described above, consider an arrangement (figure 4) relevant to physical therapy in which a circular source transducer is placed in acoustic contact with the body of a patient. Williams et al. (1987) investigated this situation and measured the temperature at the interface between the transducer and the tissue as the ultrasound was turned on and off. Using a transducer of diameter 19.1 mm, they drove it at a frequency of 5 MHz at a power level such that the temporal-average, spatial-average intensity \( I_{\text{SATA}} \) was 1.4 kW m\(^{-2} \) (0.14 W cm\(^{-2} \)) at the transducer-tissue interface. In figure 5 the individual points are values of the temperature measured by the investigators as a function of time, the ultrasound having been turned on at \( t = 0 \) and off at \( t = 180 \) s. The curves in the same figure show results of calculations employing the expression in equation (4).

![Figure 4. Schematic diagram of the arrangement used by Williams et al. (1987).](image)

The broken curve in figure 5 gives the temperature rise expected from absorption in the tissue. The tissue is assumed to be homogeneous with attenuation caused entirely by absorption, the absorption coefficient \( \alpha \) being 50 Np m\(^{-1} \) at 5 MHz (1 Np = 8.7 dB). For other quantities the following values were assumed:

- \( \rho_b = 1 \text{ g ml}^{-1} = 1000 \text{ kg m}^{-3} \)
- \( \tau = 250 \text{ s} \)
- \( c_v = c_{vb} = 4.2 \text{ J ml}^{-1} \text{ °C}^{-1} = 4.2 \times 10^6 \text{ J m}^{-3} \text{ °C}^{-1} \)
- \( K = 0.006 \text{ W cm}^{-1} \text{ °C}^{-1} = 0.6 \text{ W m}^{-1} \text{ °C}^{-1} \)

From these values and the equations given above, one obtains

\[
\kappa = 0.0014 \text{ cm}^2 \text{ s}^{-1} = 1.4 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}
\]

\[L = 4.6 \text{ mm}.
\]

Simplifying assumptions were made about the ultrasound field. Near-field complications were ignored and the beam was assumed to be projected as a right circular cylinder with intensity \( I \) given as a function of distance \( z^* \) from the transducer by

\[I = I_0 \exp(-2\alpha z^*) \tag{17}\]

\( I_0 \) being the temporal-average, spatial-average intensity at the transducer-tissue interface. Results of calculations to justify this simplification are reported elsewhere. It follows that \( q_v \), the volume rate of heat generation from absorption in the tissue, is given by

\[q_v = 2\alpha I_0 \exp(-2\alpha z^*). \tag{18}\]
Figure 5. Computed temperatures for the conditions used by Williams et al. (1987). $\alpha = 50 \text{ Np cm}^{-1}, \tau = 250 \text{ s}, I_{\text{sata}}$ at the transducer face ($I_{\text{sata}} = I_0$) is 1.4 kW m$^{-2}$ (0.14 W cm$^{-2}$). The transducer diameter is 19.1 mm. Heating at the transducer surface is 3.0 kW m$^{-2}$ (0.30 W cm$^{-2}$). The broken curve shows contribution from ultrasound absorption; the dotted curve, that from the surface heating; and the full curve, the total. Individual points are values measured by Williams et al. (1987).

The temperature at the centre of the front face of the transducer is obtained by carrying out the double integration indicated in equation (16), setting $z = 0$ and varying the radius $x$ of the ring from 0 to 9.5 mm and the distance $z^*$ from 0 to $3L$ (contributions from heat generated at greater distances having been found to be negligible).

The dotted curve in figure 5 gives the temperature produced by heat production in the transducer itself, from dielectric and acoustic dissipation. To simplify the calculations, the heat was assumed to be generated at the front face of the transducer at the rate $h$ per unit area, although it was actually generated within the transducer in some unknown spatial distribution. Under this assumption, $q, dz^*$ can then be replaced simply by $h$ in equation (16); letting $z^*$ equal zero, the equation reduces to

$$T = \frac{h}{4K} \int \frac{x}{r} \{ \} \, dx$$

where $\{ \}$ has the same meaning as in equation (4). In plotting the dotted curve in figure 5, $z$ was set equal to zero, so that $r$ reduced to $x$, and $h$ was set equal to 3 kW m$^{-2}$ (0.30 W cm$^{-2}$).

Finally, the full curve in figure 5 gives the total temperature rise on the axis at $z = 0$. It is the sum of the two other curves; that is, the sum of the heating caused by sound absorption, which occurs throughout the tissue volume, and that caused by heat generated within the transducer, which is assumed to be produced at its surface.

4. Discussion

The point source solution presented in equation (4) is useful for treating situations involving a small heat source in a perfusing medium and, by superposition, for treating continuously distributed heat sources in such a medium. For an application like that
in figures 4 and 5, the integrations required in equations (16) and (19) make only modest demands on computer time.

Theory for similar applications was presented earlier by Filipczynski (1978), in which effects of perfusion were neglected. He developed analytical solutions to equation (1) (with \( \tau \) infinite) for several special situations that are useful in gaining an insight on the temperature rise produced by ultrasound beams in soft tissues. The present theory allows some extension of the Filipczynski findings.

From the results in figure 5, obtained by using equations (4), (16) and (19), we find that the surface heating contributes much more than does the volume heating to the temperature rise at \( z = 0 \), in agreement with comments by Williams et al (1987). This is true even though the total heat production by surface heating (3 kW m\(^{-2}\) (0.30 W cm\(^{-2}\))) is only twice that produced by absorption in the tissue (1.4 kW m\(^{-2}\) (0.14 W cm\(^{-2}\))). That the temperature rise at \( z = 0 \) caused by absorption-produced heat is relatively small results from the fact that this heat is produced throughout an extended region of tissue. Distant heat sources produce a lower temperature rise than sources that are nearby, especially in the presence of perfusion. This is seen, for example, in the factor \( \exp(-r/L) \) which appears in equation (10). According to this factor (for \( L = 4.6 \) mm) perfusion causes a reduction of the steady-state temperature rise—in addition to the reduction from other causes—by a ratio of \( e^{-r/L} = 0.37 \) for each 4.6 mm increase in distance from a ring-source of heat.

It follows that the situation is reversed at points interior to the tissue. There the importance of surface heating at \( z = 0 \) is diminished relative to absorption-produced heating. In fact, the latter dominates at points deep inside the tissue.

In determining the upper curve in figure 5, alternative trial curves were calculated with various choices of parameters in order to obtain a satisfactory fit to the data. In doing this the attenuation coefficient \( \alpha \) (assumed equal to the absorption coefficient) was always given the value 50 Np m\(^{-1}\), a value within the (rather wide) range of values cited in the literature (NCRP 1983) for the tissues involved. The value for the perfusion time \( \tau \) (250 s) was determined, to a large extent, by the downward transient after \( t = 180 \) s when the ultrasound was cut off. Once \( \tau \) had been selected, \( h \) was adjusted to 3 kW m\(^{-2}\) (0.3 W cm\(^{-2}\)) to fit the data during the period just before the cut-off at \( t = 180 \) s.

It can be concluded that the theory under discussion, based on equations (4), (16) and (19), is reasonably consistent with the experimental observations in figure 5. However, fitting the theory to these data does not, in itself, provide an unambiguous determination of the parameters. Other experiments would be needed in order to obtain accurate values of \( \alpha \), \( \tau \) and \( h \).

While equation (1), a linear equation, appears to be useful in accounting for the main aspects of such results as those in figure 5, it lacks features that would be needed for other situations. For example, it does not take into account changes of perfusion 'constants' with temperature. Neither is it adequate for dealing with the influence of temperature gradients which exist close to the surface of a living body. Further development of bio-heat transfer equations is needed to treat these and other complications.

Acknowledgments

The author is grateful to Dr D L Miller, Dr A R Williams and Professor W Swindell for helpful comments on the analysis presented here and also to Drs Miller and
Williams for providing the original data for figure 5. This work was supported, in part, by the US National Institute of Health via Grant CA 42947.

Résumé

Solutions de l'équation du transfert biothermique.

L'auteur présente et discute une solution de l'équation de transfert biothermique pour une source ponctuelle. A partir de cette solution de base, il est possible, en principe, de déterminer la distribution des températures résultant d'une distribution quelconque de sources de chaleur en effectuant des superpositions. A titre d'exemple, l'auteur utilisa cette méthode pour calculer la température à la surface du corps à l'endroit où le faisceau thérapeutique d'ultrasons est appliqué. Il effectue une comparaison de ces données à des résultats expérimentaux publiés récemment par Williams et collaborateurs.

Zusammenfassung

Lösungen der Biowärmetransfergleichung.


References

Morse P M and Feshbach H 1953 Methods of Theoretical Physics (New York: McGraw-Hill) section 12.1, equation (12.1.20)
NCRP 1983 Biological Effects of Ultrasound: Mechanisms and Clinical Implications, Report No 74 (Bethesda, MD: NCRP)
Pennes H H 1948 J. Appl. Physiol. 1 93-122
Swindell W 1984 Br. J. Radiol. 57 167-8