THE RADIATION FIELD

1.1 Introduction

The term field in mathematical physics designates a physical quantity which is defined as a function of position ***r***. Consider a radiation monitor, which for the present is defined as a device which registers a response R via a meter deflection or number on a digital display, quantifying in some way the radiation interacting with it. A radiation survey consists of determining a set of readings $\{R_i, i=1,N\}$ at a set of corresponding positions $\{\boldsymbol{r}_i, i=1,N\}$. It is well-known from the sampling theorem that if the readings are taken at sufficiently small intervals compared to the response variation, that it is possible with appropriate interpolation to reconstruct the continuous function $R(\boldsymbol{r})$, over the region surveyed. This function constitutes a scalar field. It may or may not be a useful description of the radiation field, depending upon the application. As we will see, it is certainly an incomplete description of the field.

The physical quantity that is accepted as an appropriate descriptor of the radiation field involves the concept of fluence, a word that is not widely used elsewhere. Fluent is the ability of a substance to flow. In fact the concept of fluence, if not the word, is commonly expressed in the daily weather forecast when it is announced that 5 mm of rain is expected overnight. Now this is not the normal way to express the quantity of a fluid, as anyone who buys a litre of milk knows. So why doesn't the forecaster simply quote the total volume of rain that will fall? Well, in the first place the boundaries of the region over which the rain will fall are poorly defined. And in the second place no one is interested in the rain falling on the fields of a farm in the next county. If one sets out a container of area A, then the volume collected for a rainfall t is V=At. To put it the other way round, the volume of rain that fell per unit area is t=V/A. Thus the forcaster could have announced that 0.5 cm^3/cm^2 of rain is expected. This could also to a good approximation be given in terms of mass as 0.5 g/cm^2, or in chemical terms 27.8 mM/cm^2. Ultimately in the most fundamental terms this corresponds to 1.67×10^{22} $molecules/cm^2$. It is this most basic description that is used in radiation physics so the SI unit of fluence is m^{-2}.

One could now consider performing a fluence survey by setting out a large number of containers at positions $\{\boldsymbol{r}_i, i=1,M\}$ and determining for each the number N_i of molecules received. The fluence would then be given by $< \Phi >_i = N_i/A_i$ where A_i is the area of the container at position $\boldsymbol{r}_i$. If as is implied the fluence can vary continuously with position, then this measurement would only provide us with an estimate averaged over the collection area. The measurement will only be unambiguous if the collection area is sufficiently small that there is no variation of the fluence over the points encompassed. If this were the case then the fluence field $\Phi(\mathbf{r})$ could be determined.

The amount of rain collected is of course the time integral of the rate at which the rain accumulated. If the rainfall lasted 10h then a simple calculation shows that water molecules were accumulated at an average rate of 5×10^{17} $cm^{-2}s^{-1}$. This corresponds to the average fluence rate. Of course if the rainfall isn't steady then the instantaneous rate must be obtained by taking the limit of measurements made with small time intervals about the time of interest. An alternative terminology is used for fluence rate. The correct form of this terminology, consistent with electromagnetic theory and historical usage is flux density. Flux was originally used for the total rate of flow of a quantity. For flowing charge then the term flux would refer to the current. This is also consistent with Gauss's law, where flux corresponds to the integral of the field over area. However, the term flux is often used, incorrectly in my opinion, when flux density is intended. This confusion is avoided with the term fluence rate.

The question now arises as to whether or not the fluence rate determined over all spatial positions would provide a complete description of the field. A little thought would indicate the answer to be in the negative. Consider for the moment a quiet room. If our sensory perceptions were more acute, the room and in particular the space between the walls would appear anything but quiet. There would of course be the spectacle of the nitrogen and oxygen molecules busily jostling one another in a random dance. More spectacular however would be the sudden emergence from one wall of a 2.6 MeV γ-ray, which shoots across the room, collides with the far wall, and re-emerges as a 0.25 MeV γ-ray moving back upon itself. It then collides with a nitrogen molecule in the centre of the room, deflecting from its path before disappearing into the original wall. As a result of the collision, a 70 keV electron is dislodged from the nitrogen molecule which undergoes a zig-zag motion colliding over and over again with nitrogen and oxygen molecules leaving a debris of ionized, excited and dissociated species in its wake. And then a 400 MeV mu meson streaks past, moving in virtually a straight line, again leaving ionized molecules as a record of its track.

The above illustrates that a description of the field should include an indication of the kinds of radiation, the energy and direction of motion as well as position. In this regard it is important to note that the energy used as a descriptor does not include rest mass energy. From this viewpoint it would be sufficient to list the number of particles or photons with a certain energy moving in a certain direction in the room at any time, in the manner of a histogram with energy and direction bins. That is, the number is distributed over energy and direction. On the astronomical scale , the dimensions of the room are negligible, so presence in the room identifies a unique location. In order to preserve this attribute at any desired scale, it is necessary to define a volume sufficiently small with the scale of the measurement. Thus for example on a meter scale a cubic millimetre volume could be located with negligible uncertainty in its coordinates. By and large, one would naturally expect the number in a given region to be proportional to the volume so a more fundamental quantity is the ratio of number to volume, the number density.

1.2 The number density

The radiation field consists of a large number of particles or quanta distributed in space and time moving in various directions with differing energies. A description of such a system is necessarily statistical in nature and rests on the introduction of a six-dimensional phase space, the direct product of configuration space and momentum space constructed from the position and momentum co-ordinates. The number of particles in a volume element of phase space at time t is then written as

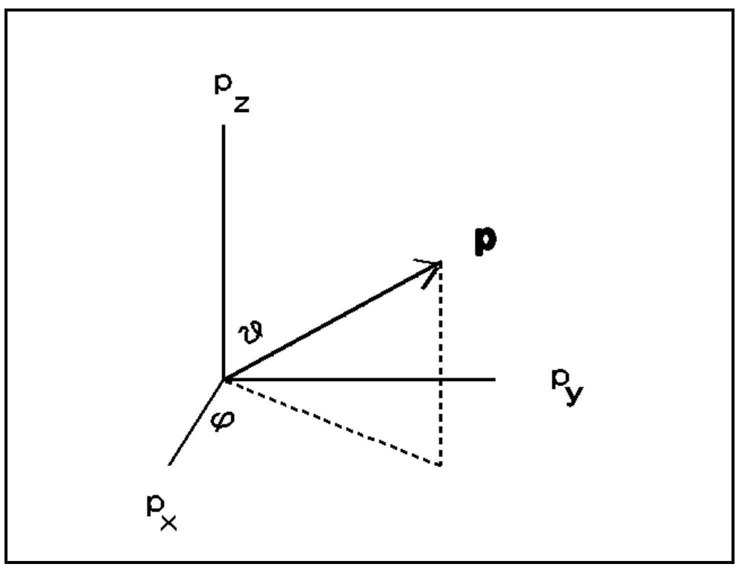

Figure 1. **Momentum Space**

$$d^6N=\bar{n}(\boldsymbol{r},\boldsymbol{p},t)d^3\boldsymbol{r}d^3\boldsymbol{p} \qquad \textbf{(1)}$$

While the momentum is the fundamental quantity used in classical statistical mechanics it is advantageous to reformulate the description in terms of the more useful variables of kinetic energy and direction. This may be accomplished by introducing a spherical polar co-ordinate system in momentum space so that

$$\begin{aligned} p_x &= p\ \sin\theta\ \cos\varphi \\ p_y &= p\ \sin\theta\ \sin\varphi \\ p_z &= p\ \cos\theta \end{aligned} \qquad \textbf{(2)}$$

or, more concisely,

$$\boldsymbol{p} = p\boldsymbol{\Omega} \qquad \textbf{(3)}$$

where the unit vector **Ω** with components defined in equation (2) is oriented along the direction of motion.

Transformation to energy is accomplished using the relationship to momentum

$$W^2 = p^2\ c^2 + m_0^2\ c^4 \qquad \textbf{(4)}$$

where the total energy W is given by

$$W = E + m_0c^2 \qquad \textbf{(5)}$$

The volume element in momentum space can now be written

$$\begin{aligned} d^3\boldsymbol{p} &= p^2dp\sin\theta d\theta d\varphi \\ &=(pW/c^2)dEd^2\Omega \end{aligned} \qquad \textbf{(6)}$$

The angular number spectral density is defined as

$$\tilde{n}(\boldsymbol{r},\boldsymbol{E},t) = \bar{n}(\boldsymbol{r},\boldsymbol{p},t)pW/c^2 \qquad cm^{-3}MeV^{-1}st^{-1}$$

so that

$$d^6N = \tilde{n}(\boldsymbol{r},\boldsymbol{E},t)\ d^3\boldsymbol{r}d^3\boldsymbol{E} \qquad \textbf{(7)}$$

In equation (7) the vector **E** = E **Ω** is oriented along the propagation direction and has magnitude equal to the kinetic energy. The differential $d^3\mathbf{E} = dEd\Omega$. The vectorial energy is introduced as a compact notation and is not meant to imply that energy is a vector in the physical sense. The number density, $\hat{n}(\boldsymbol{r},t)$ results from integration of the angular number spectral density over energy and direction, ie $d^3\boldsymbol{E}$

2. The angular fluence rate

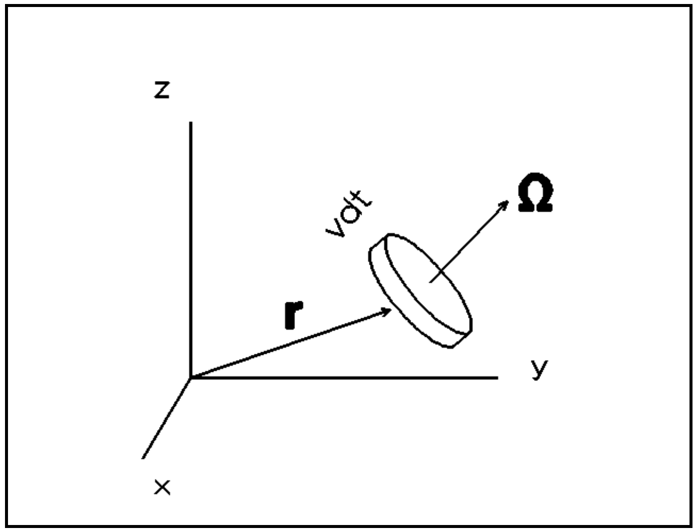

Figure 2. **Particle Flow**

The motion of the particles introduces the concept of a flow variable. Perhaps it is best to first consider a simple everyday example-traffic flow. Although in times of congestion it might not seem like it, there is a difference between describing cars standing in a parking lot and those on a road. In the former case, one could describe the situation by how closely they are crowded together, analogous to density. In describing the situation of traffic flow however, a different approach is taken. The traffic engineer lays a tube across the road forming a pneumatic indicator. After a certain time has elapsed, the number of vehicles which have crossed the tube are noted. The ratio of this number to the time is an estimate of the flow rate. Now obviously the flow rate for a 3-lane wide highway cannot be directly compared with that of a one-lane highway. The meaningful parameter is the flow rate per lane. This is analogous to the fluence rate. Notice that both in order to make the measurement and to define the concept, it is necessary to introduce a boundary-the tube- separating two regions of space-the road to the left and the road to the right. The measurement then involves a tally of the vehicles which cross the boundary thereby transferring from one region to the other.

The phenomenon of flow is most easily described in a one- dimensional momentum space. In this case all particles are characterized by the same direction vector, a situation which occurs for ideal fluid flow in a pipe. A radiation field with this characteristic is referred to as a perfectly collimated beam. It will also be assumed that all the particles have the same velocity. Such a radiation field is monoenergetic. The mono-energetic, unidirectional radiation field forms a basis for more general structures. The angular number spectral density then takes the form

$$\tilde{n}(\boldsymbol{r},\boldsymbol{E}) = \hat{n}\delta(1-\boldsymbol{\Omega}\cdot\nabla z)\delta(E-E_0) \qquad \textbf{(8)}$$

describing a monoenergetic field with energy E_0 in which all particles move exactly in the z-direction. These particles will all have the same velocity v∇z. In 3 dimensions the boundary between two regions is an area.

In time interval dt the volume element $d^3\mathbf{r} = da_\perp vdt$ is generated by the set of all particles with velocity v passing the cross sectional area $da_\perp$. The quantity

$$\varphi = \hat{n}v, \qquad \textbf{(9)}$$

where the velocity

$$\boldsymbol{v} = \boldsymbol{p}c^2/W, \qquad \textbf{(10)}$$

represents the number of particles crossing a unit area of beam cross-section per unit time, where n is the density of particles . The product in equation (9) is in the case of a beam the particle fluence rate. Note that the direction of motion is perpendicular to the plane of the cross-sectional area.

The situation is less straightforward for the general radiation field in which particles are moving with differing directions and energies. In this case the field is decomposed into beams reflecting the particle distribution with respect to direction. A quantity analogous to that in equation (9), referred to as the angular fluence rate spectrum is defined as

$$\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t) = \tilde{n}(\boldsymbol{r},\boldsymbol{E},t)\cdot v \quad cm^{-2}MeV^{-1}st^{-1}s^{-1} \qquad \textbf{(11)}$$

The angular fluence rate spectrum corresponds to the number of particles in a specific energy and directional group passing a unit perpendicular area per second at time t. The group structure corresponds to specifying an energy range E→ E+dE and a directional range described by the solid angle element $d^2\boldsymbol{\Omega}$ about $\boldsymbol{\Omega}$. Note that the orientation of the area passed is such that the normal to that area is $\boldsymbol{\Omega}$. The angular fluence rate spectrum is proportional to the particle probability density function for the distribution with respect to energy and direction. The situation of a well-collimated beam, also referred to as a monodirectional radiation field is mathematically singular in nature and requires the introduction of Dirac delta functions as in Eqn(8). This requirement also applies to the idealization of a monoenergetic spectrum. In optics the angular fluence rate spectrum is referred to as the photon spectral radiance.

3. Integrated quantities

While the angular fluence rate spectrum provides a complete description of the radiation field such detail is often unnecessary. For example the direction of motion of the particle is irrelevant in interacting with an atom. For this reason the quantity obtained by integration over direction

$$\varphi(\boldsymbol{r},E,t) = \int \tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)d^2\boldsymbol{\Omega} \;\; MeV^{-1}cm^{-2}s^{-1} \qquad \textbf{(12)}$$

is commonly used and is referred to as the fluence rate spectrum. Note that an element of fluence rate, $\hat{\varphi}(\boldsymbol{r})$, is given by

$$d\hat{\varphi} = \tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)\;d^3\boldsymbol{E} \qquad \textbf{(13)}$$

The integration in equation (12) and the resultant fluence rate spectrum must be interpreted with care because the defining area must re-orient as the integration over direction proceeds. The simplest approach is to select a circular area for each angular fluence rate in which case the object generated by reorientation over all directions is a sphere. The fluence rate spectrum is then the number of particles in a specified energy group that cross a sphere of unit cross- sectional area in a unit time. The fluence rate results from integration of the spectrum over energy. Integration over a specified time period leads to the quantity chosen as fundamental by the I.C.R.U., the fluence spectrum given by

$$\Phi(\boldsymbol{r},E) = \int \varphi(\boldsymbol{r},E,t)\;dt \quad MeV^{-1}cm^{-2} \qquad \textbf{(14)}$$

Note that it is convenient to define an angular fluence spectrum as a time integral of the angular fluence rate spectrum

$$\tilde{\Phi}(\boldsymbol{r},\boldsymbol{E}) = \int \tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)\ dt \quad MeV^{-1} cm^{-2} st^{-1} \qquad \textbf{(15)}$$

The fluence is also identified as the density of track length laid down during the exposure period. From equ.(13)

$$d\Phi = \tilde{n}(\boldsymbol{r},\boldsymbol{E},t)\ d\Omega \cdot v dt \qquad \textbf{(16)}$$

Since vdt=ds is an element of the particle path or track and integration over solid angle yields the number density of particles of a given group the relation between fluence and track length follows via

$$\Phi = \int n(\boldsymbol{r},E,t) ds \qquad \textbf{(17)}$$

If the angular fluence, or fluence rate is independent of the directional coordinate $\boldsymbol{\Omega}$, the radiation field is said to be isotropic. In the simple case of an isotropic field at $\boldsymbol{r}$, $\varphi(\boldsymbol{r},E,t) = 4\pi\tilde{\varphi}(\boldsymbol{r},E,t)$. Integration of the fluence spectrum over energy gives the fluence, $\hat{\Phi}(\boldsymbol{r})$

4. Vectorial quantities

A critical aspect in the definition of angular fluence is the perpendicular area element. In integrating over all directions in Eqn(12), the area is allowed to rotate. Of course it may be important to analyse situations involving an area of fixed orientation. Then an area element vector d**A** may be defined as the product of the area with a normal unit vector. For a directional group making angle θ with the normal the perpendicular area satisfies

$$dA_{\perp} = dA\cos\theta = \boldsymbol{\Omega} \cdot d\boldsymbol{A} \qquad \textbf{(18)}$$

The above relation can then be incorporated into each directional group by defining a new quantity, the angular current density spectrum. Again, the term spectrum will generally be understood.. Fundamentally the angular current density is given by

$$\boldsymbol{j}(\boldsymbol{r},\boldsymbol{E}) = \tilde{n}(\boldsymbol{r},\boldsymbol{E})\boldsymbol{v} \qquad \textbf{(19)}$$

While the angular fluence rate spectrum is defined in terms of the magnitude of the velocity and hence is a scalar quantity the angular current density spectrum, which is deceptively similar in formula appearance, is a vector quantity. In order to emphasize this difference it may be defined as

$$\boldsymbol{j}(\boldsymbol{r},\boldsymbol{E},t) = \tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t) \cdot \boldsymbol{\Omega} \qquad \textbf{(20)}$$

In this case integration over all directions leads to the vector

$$\boldsymbol{J}(\boldsymbol{r},E,t) = \int \boldsymbol{j}(\boldsymbol{r},\boldsymbol{E},t) d\boldsymbol{\Omega} \qquad \textbf{(21)}$$

referred to as the current density spectrum. Then the quantity $\boldsymbol{J}(\boldsymbol{r},E,t) \cdot d\boldsymbol{A}$ represents the net number of particles in an energy group of the radiation field which cross an area element per unit time. In general the term spectrum for the above quantities may be left implicit. The current density is fundamental in describing radiation transport. The integral of the current density over time is referred to as the vector fluence, **Φ.** Let the boundary be a plane with area element $d\mathbf{A}=dA\nabla z$. Then the net number of particles of a given energy group crossing the plane per unit area is the z-component of the vector fluence. The z-component of the vector fluence is referred to as the planar fluence, Φ_P. The planar fluence is particularly useful in those situations in which there is lateral symmetry so that the radiation field is independent of the azimuthal directional coordinate. It can be shown that in this case the other components of the vector fluence vanish. Two partial components of the planar fluence can be defined as the half integrals

$$\Phi_+(\boldsymbol{r},E) = \left| \int_{\theta=0}^{\theta=\pi/2} \int_{\varphi=0}^{\varphi=2\pi} \tilde{\Phi}(\boldsymbol{r},\boldsymbol{E})\cos\theta\, d\boldsymbol{\Omega} \right|$$
$$\Phi_-(\boldsymbol{r},E) = \left| \int_{\theta=\pi/2}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \tilde{\Phi}(\boldsymbol{r},\boldsymbol{E})\cos\theta\, d\boldsymbol{\Omega} \right| \tag{22}$$

These correspond to the number crossing the plane per unit area in the positive and negative directions respectively. The planar fluence, the net number crossing, is the difference between the positive and negative components. The total number crossing the plane per unit area is the sum of these two components.

An important quantity is the energy flow vector given by

$$\boldsymbol{g}(\boldsymbol{r},t) = \int E\cdot \boldsymbol{J}(\boldsymbol{r},E,t)\,dE = \int \boldsymbol{E}\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)\,d^3\boldsymbol{E} \tag{23}$$

and its time integral **G**(**r**). In particular for a region of space $\mathbb{R}$ bounded by a surface $\sum$, the surface integral

$$R = -\oint_{\Sigma} \boldsymbol{G}(\boldsymbol{r})\cdot d\boldsymbol{a} \tag{24}$$

represents the net amount of radiant energy transferred to $\mathbb{R}$. Note that by convention the area element vector for a closed surface is directed along the outward pointing normal.

5. Measurements

The measurement of the angular fluence spectrum is uncommon and difficult. An ideal arrangement is sketched in Fig.3. The shaded area, a disc of radius a, represents the entrance window of a detector. A collimator defines the conical portion shown, opening to a disc of radius b=d/2 at the oblique distance R. The system is axially symmetric about the normal at the centre of the discs. The centre of the shaded disc is at the point $\boldsymbol{r}$. A local coordinate system may be used with the z-axis anti-parallel to $\boldsymbol{\Omega}$ and the origin at the centre of the shaded disc. A point on this disc may then be specified by polar coordinates (ρ,α) ,or $\boldsymbol{\rho}$ and the particle direction by the standard spherical polar coordinates (θ,φ) defining the vector $\boldsymbol{\omega}$.

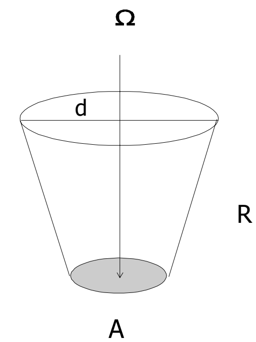

Figure 3. Angular Fluence.

It is assumed that the detector is shielded so that no radiation is detected moving with a positive z-component. The correct expression for the observed counting rate is then

$$\frac{dN(E,t)}{dt}=\int_{\rho=0}^{a}\int_{\alpha=0}^{2\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\theta_0(\rho,\varphi)}\varepsilon(\boldsymbol{\rho},\boldsymbol{\omega},E)\tilde{\varphi}(\boldsymbol{\rho},\boldsymbol{\omega},E,t)\cos\theta\, d\omega\,\rho\, d\rho\, d\alpha \qquad (25)$$

where $\varepsilon\,(\boldsymbol{\rho},\boldsymbol{\omega},E)$ is the detector efficiency. If the detector dimensions are large with respect to the radius a then the efficiency will be independent of the position and direction of the incident particle. Also if the angular fluence rate does not very substantially over the shaded area, or over the range of directions allowed by the collimation then the angular fluence rate is constant over the region of integration. Moreover, in order for this to be true, all polar angles must satisfy $\theta \ll 1$, so $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. The expression then simplifies to

$$\frac{dN(E)}{dt}=\pi\varepsilon(E)\tilde{\varphi}(\boldsymbol{r},\boldsymbol{\Omega})\int_{\rho=0}^{a}\int_{\varphi=0}^{2\pi}\theta_0^2(\rho,\varphi)\,d\varphi\,\rho\, d\rho \qquad (26)$$

The exact expression for θ_0 is quite complicated. In the case where a<<d, it becomes

$$\theta_0^2=\frac{d^2}{4R^2}\left[1-\frac{4\rho}{d}\cos\varphi\right] \qquad (27)$$

Since the second term does not contribute to the integral, the event rate becomes

$$\frac{dN(E,t)}{dt} = (\pi a^2)(\frac{\pi d^2}{4R^2})\varepsilon(E)\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t) \qquad (28)$$

The interpretation of Eqn(28) is straightforward. For everything small enough, the first term in brackets is the area perpendicular to the direction, and the second factor is the solid angle representing the range of directions accepted.

The angular fluence is then determined from the event rate, the dimensions and the detector efficiency. The determination of this last quantity is no easy task. Angular fluences are determined in cosmic ray counter telescope arrangements in which the collimation is implicit.

If the detector collimation is removed so the detector responds with equal efficiency to incoming particles of all directions in a hemisphere then the device can be used to measure a current density component as the difference between the event rates observed with the detector oriented parallel to and antiparallel to the component direction.

6. Radiometric Quantities

The description of light was developed in the 19^{th} century before the concept of the photon evolved. As a result the classical radiometric quantities used to describe the radiation field emphasize field energy and wavelength distributions. The fundamental quantity is the spectral radiance which is related to the angular fluence rate spectrum by

$$L_\lambda(\boldsymbol{r},\boldsymbol{\Omega}) = \frac{h^2c^2}{\lambda^3}\tilde{\varphi}(\boldsymbol{r},\frac{hc}{\lambda},\boldsymbol{\Omega}) \qquad (29)$$

The units usually used are the somewhat redundant $W \cdot m^{-2} \cdot sr^{-1} \cdot nm^{-1}$. The term preceding the angular fluence is the product of the energy with its derivative with respect to wavelength. The spectral radiance is an explicit function of wave length, but the dependence is implied by the subscript.

The spectral irradiance corresponds to the radiant flux per unit area per unit wavelength incident on or emerging from a point on a specified surface. The two cases are contained in

$$E_\lambda(\boldsymbol{r}) = \frac{h^2c^2}{\lambda^3}\frac{d\Phi_\pm}{dt} \tag{30}$$

where the arguments of $\Phi_\pm(\boldsymbol{r},E,t)$ are understood and the units become $W{\cdot}m^{-2}{\cdot}nm^{-1}$. The spectral radiant flux is the integral of the spectral irradiance over area with units of $W{\cdot}nm^{-1}$. The accepted symbol for radiant flux is identical to that used by convention in ionizing radiation for fluence. An additional quantity is spectral intensity defined as

$$I_\lambda = \int L_\lambda dA_\perp = \int L_\lambda \cos\theta d^2\boldsymbol{r} \tag{31}$$

with units of $W{\cdot}sr^{-1}$.

7. Streaming

A particularly simple case is that in which the particles of the field move in free space. In this situation each particle moves in a straight line, if the action of the other field particles may be neglected. A common approximate situation is that of the rays of light streaming down from the sun. Of course when this phenomenon is actually observed in the earth's atmosphere, the free case condition is violated and a fraction of the trajectories is altered, deflecting the light to our eyes. For the ideal case, a beam of particles of a given energy group will remain undisturbed. It would seem reasonable then that the angular fluence spectrum would be invariant along the beam direction.. This is developed in more detail by considering the relation between the particles leaving one region of space and arriving at another. Since the particle trajectories are unaltered every particle leaving the first region in the appropriate direction will arrive at the second region. We restrict discussion to infinitesimal regions so that the angular fluence spectrum may be taken as constant over the small range of spatial and directional coordinates.

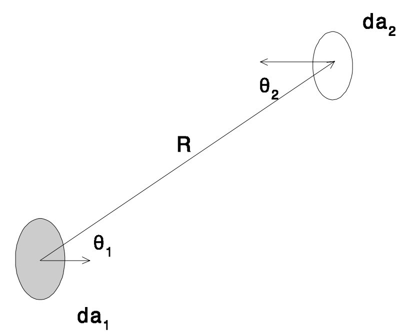

Figure 4. Streaming geometry

In the drawing shown the region from which the particles leave has area da_1 oriented at an angle θ_1 with respect to the particle trajectories which lead to the receiving region, a distance R away in the direction $\mathbf{\Omega}$. In order that the particle strike the receiving region it must be travelling within the solid angle range $d\Omega_2$ subtended by da_2 , oriented at the angle θ_2 with respect to the trajectory at the distance R. Then the solid angle is $d\Omega_2 = da_2 \cos\theta_2/R^2$. The area perpendicular to $\mathbf{\Omega}$.is $da_1\cos\theta_1$. If the number of particles leaving the first region in the appropriate range of direction is designated dN, then the angular fluence spectrum at this point satisfies

$$\tilde{\Phi}_1 = \frac{R^2 dN}{da_1 \cos\theta_1 da_2 \cos\theta_2 dE} \qquad \textbf{(32)}$$

To calculate the angular fluence spectrum at the receiving position note that all trajectories received must lie in the directional range $d\Omega_1 = da_1\cos\theta_1/R^2$ around $\mathbf{\Omega}$ and the perpendicular area is $da_2\cos\theta_2$. Thus since the number of particles arriving is also dN the angular fluence spectrum at the second position is

$$\tilde{\Phi}_2 = \frac{R^2 dN}{da_2 \cos\theta_2 da_1 \cos\theta_1 dE} = \tilde{\Phi}_1 \qquad \textbf{(33)}$$

Since the two regions are arbitrary this demonstrates the invariance. The number of particles incident on the receiving region per unit energy interval can then be written

$$N(E) = \int\int \tilde{\Phi} \frac{\cos\theta_1 \cos\theta_2}{R^2} da_1 da_2 \qquad \textbf{(34)}$$

Note that the equation is symmetrical in the coordinates of both regions. This is indicative of a reciprocity relation, which is reasonable on physical grounds. The same equation can be used to calculate the number of particles incident on region1 which have left region 2. In effect, this is equivalent to reversing the directions of all the particle trajectories. In the case when the two regions have vanishingly small dimensions then all the terms in the integrand become constant. The total number of particles leaving region 1 per unit energy and solid angle in direction $\boldsymbol{\Omega}'$ may be written $\tilde{Q}(\boldsymbol{r}',\boldsymbol{E}') = \tilde{\Phi}(\boldsymbol{r}',\boldsymbol{E}')\cos\theta_1 \Delta a_1$**,** where $\boldsymbol{r}'$ is the position of region1. These particles then approximately emanate from the point in space designated as $\boldsymbol{r}'$. Then the number of particles per unit energy at the receiving position $\boldsymbol{r}$ is given by

$$\Delta N(E) = \frac{\tilde{Q}(\boldsymbol{r}',\boldsymbol{E})}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\cos\theta_2 \Delta a_2 \qquad \textbf{(35)}$$

All these particles are essentially moving in the direction $\boldsymbol{\Omega}$ joining the point $\boldsymbol{r}'$ to $\boldsymbol{r}$, so the fluence at the receiving point may be written

$$\tilde{\Phi}(\boldsymbol{r},\boldsymbol{E}') = \frac{\tilde{Q}(\boldsymbol{r}',\boldsymbol{E}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\delta(1 - \mathbf{\Omega}\cdot\mathbf{\Omega}') \qquad \textbf{(36)}$$

In this result we see the famous $1/R^2$ law. The region from which the point emanates is equivalent to a point source. If the quantity $\tilde{Q}$ is independent of direction then the point source is isotropic and the total number of particles emitted is $4\pi\tilde{Q}$ Note that an isotropic point source does not produce in general an isotropic radiation field. In fact in the case of streaming under discussion the field at any point is unidirectional along the line joining the point to the source, the most anisotropic situation possible. The isotropic point source under streaming is a commonly used first order approximation in many cases. The fluence at any point is merely the number of particles emitted divided by the surface area of the sphere with radius equal to the distance from the source at which the fluence is to be evaluated. The vector fluence is directed radially and the radial component is the same as the fluence.

An example which illustrates the usefulness of the angular fluence invariance is the problem of determining the fluence at any point due to streaming from an infinite plane. For the fluence one postulates the existence of a small sphere of cross sectional area a_c at a distance z from the sphere. All points that are situated at distance R from the sphere lie on a ring in the plane. Choosing the z-axis to go through the sphere the radius of the ring ,ρ, satisfies $\rho^2 = R^2 - z^2$ and hence $\rho d\rho = RdR$. Then assuming azimuthal symmetry, the number passing through the sphere can be written

$$N = 2\pi \int_0^\infty \tilde{\Phi} \frac{a_c}{R^2} \cos\theta R dR \quad \textbf{(37)}$$

where θ is the angle between the line of join and the z-axis so that z=Rcosθ. Remembering that the fluence is the ratio of N to a_c gives

$$\Phi = 2\pi \int_0^{\pi/2} \tilde{\Phi} \sin\theta d\theta \quad \textbf{(38)}$$

Of course this is simply the definition of fluence if the angular fluence is interpreted as existing at the point of interest rather than on the plane. If the angular fluence is isotropic one arrives at $\Phi = 2\pi\tilde{\Phi}$ a physically obvious result. Similar considerations show that the planar fluence is half the fluence in this case.

Summary

Angular number density spectrum : $\tilde{n}(\boldsymbol{r},\boldsymbol{E},t)=\dfrac{d^6N}{d^3\boldsymbol{r}d^3\boldsymbol{E}}$

Number density spectrum: $n(\boldsymbol{r},E,t)=\dfrac{d^4N}{d^3\boldsymbol{r}dE}=\int\limits_{4\pi}\tilde{n}(\boldsymbol{r},\boldsymbol{E})d^2\boldsymbol{\Omega}$

Number density: $\hat{n}(\boldsymbol{r},t)=\dfrac{d^3N}{d^3\boldsymbol{r}}=\int\limits_0^\infty n(\boldsymbol{r},E,t)dE=\int\limits_{E=0}^{\infty}\int\limits_{\Omega=4\pi}\tilde{n}(\boldsymbol{r},\boldsymbol{E},t)d^3\boldsymbol{E}$

Angular number density: $n^{\Omega}(\boldsymbol{r},\boldsymbol{\Omega},t)=\dfrac{d^5N}{d^3\boldsymbol{r}d^2\boldsymbol{\Omega}}=\int\limits_0^\infty\tilde{n}(\boldsymbol{r},\boldsymbol{E},t)dE$

Velocity: $\boldsymbol{v}=v\boldsymbol{\Omega}$

Perpendicular area: $d^2\boldsymbol{A}_\perp=d^2A_\perp\Omega$

Configuration space volume element: $d^3\boldsymbol{r}=vdtd^2A_\perp$

Angular fluence rate spectrum: $\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)=\dfrac{d^6N}{dA^2_\perp d^3\boldsymbol{E}dt}=\tilde{n}(\boldsymbol{r},\boldsymbol{E},t)v$

Fluence rate spectrum: $\varphi(\boldsymbol{r},E,t)\equiv\int\limits_{4\pi}\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)d^2\Omega=n(\boldsymbol{r},E,t)v$

Fluence rate: $\hat{\varphi}(\boldsymbol{r},t)\equiv\int\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)d^3\boldsymbol{E}=\int\varphi(\boldsymbol{r},E,t)dE$

Angular fluence spectrum: $\tilde{\Phi}(\boldsymbol{r},\boldsymbol{E})=\int\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)dt$

Current density spectrum: $\boldsymbol{J}(\boldsymbol{r},E,t)=\int\limits_{4\pi}\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)\boldsymbol{\Omega}d^2\Omega$

Vector fluence spectrum: $\boldsymbol{\Phi}(\boldsymbol{r},E)=\int\limits_{4\pi}\tilde{\Phi}(\boldsymbol{r},\boldsymbol{E})\boldsymbol{\Omega}d^2\Omega=\int\boldsymbol{J}(\boldsymbol{r},E,t)dt$

Energy flow vector: $\boldsymbol{g}(\boldsymbol{r},t)=\int\tilde{\varphi}(\boldsymbol{r},\boldsymbol{E},t)\boldsymbol{E}\boldsymbol{d}^{\boldsymbol{3}}\boldsymbol{E}$

Time integrated energy flow vector: $\boldsymbol{G}(\boldsymbol{r},t)=\int\tilde{\Phi}(\boldsymbol{r},\boldsymbol{E},t)\boldsymbol{E}\boldsymbol{d}^{\boldsymbol{3}}\boldsymbol{E}=\int\boldsymbol{g}(\boldsymbol{r},t)dt$

PROBLEMS FOR DISCUSSION
(RADIATION FIELD)

1. (a) For what radiation fields would you expect the shape of the number density spectrum to be the same as the fluence rate spectrum.

   (b) Find a relationship between current density and fluence rate. For what condition are they equal?

   (c) Sketch the angular fluence for a field produced for a broad perfectly collimated beam incident upon a perfect mirror at a point between the light source and the mirror as a function of angle with respect to the normal to the mirror, which may be taken as the z-axis. Discuss the case of diffuse reflection.

2 A detector with a 1cm diameter entrance window is embedded in a lead collimator for which the entrance aperture is also 1cm, located 15 cm from the detector face. The axis of the system, pointing from the detector face to the collimator aperture is the z-axis. The detector efficiency for the energy region of interest is 1%. The true counting rate, (ie corrected for dead time and detector energy response), for photons in a 3 keV energy window and a 70 keV lower level is 2000 cpm. What is the value of the angular fluence rate spectrum under these conditions? What are the independant variables to be specified?

3. A radiation field has azimuthal symmetry if the angular fluence is independent of the orientation about a specific direction. For convenience this direction is taken as the z-axis, so the **directional** coordinates representing **Ω** are the usual angles θ and φ. The angular fluence rate then is independant of φ. In this case the angular fluence spectrum can always be expanded in Legendre polynomials according to

$$\tilde{\Phi}(\boldsymbol{r},\boldsymbol{E}) = \sum_{\ell=0}^{\infty} \Phi_\ell(\boldsymbol{r},E) P_\ell(\cos\theta)$$

Find the fluence rate and current density spectra in the above case. Remember that the Legendre polynomials satisfy

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

4. The invariance property of angular fluence can be extended to the case of transparent optical media.

Consider the case of a boundary with specular reflection in which the rays obey Snell's law. Starting with the definition

$$\tilde{\Phi} = \frac{dN}{dA_{\perp} d\Omega}$$

for a monochromatic light field, find the relation between the angular fluence in the incident medium, index of refraction n_i, to that in the refracted medium, n_r, assuming the fraction of photons transmitted is τ. For perfect transmission, what would be the invariant quantity?. Hint: You will find both Snell's law and the relation obtained from the differential of Snell's law useful.

5. The angular fluence rate for a monoenergetic neutron field is given by

$$\tilde{\Phi}(\boldsymbol{r},\boldsymbol{\Omega}) = \frac{\alpha C}{2 r \sinh\alpha} e^{\alpha\cos\theta}$$

where C, and α are constants, r is the distance from the origin and θ is the angle made by the neutron velocity with the z-direction.

(a) Find the fluence rate at r.

(b) Find the volume integral of the fluence rate over a very thin disc of radius a, and thickness t situated with its axis on the z-axis at a distance z_0 with $z_0 >> t$.

(c) Find the current density at r.

(d) Find the number of particles passing through the face of the above disc.

6. A container of volume V is fitted with an orifice of area A. The container is filled to a guage pressure of P(0) with the orifice sealed. At t=0 the orifice is opened. Find an expression for the pressure as a function of time, given that the molecules move randomly in direction with velocity v (average).

7. The thermal neutron fluence rate in the core of a high flux reactor is of the order of 10^{16} cm $^{-2}$ s^{-1}. Assume a temperature such that $kT=1/40\ eV$ where k is Boltzmann's constant. Calculate the pressure

exerted by the neutrons taken as an ideal gas.

8. A monoenergetic radiation field has an angular fluence rate which depends only upon the polar angle θ

(a) What property does this imply about the current density?

(b) Find a relation between the average of cosθ and appropriate field quantities.

(c) In electron beam radiation therapy a highly collimated beam impinges upon the patient. Due to scattering the beam diverges as the depth of penetration increases. What happens to the flux density?

9. (a) Find the total number of particles which pass through a disc of radius R centred on the z-axis oriented in the x-y plane. The angular fluence rate is

$$\tilde{\varphi}(\boldsymbol{r},\boldsymbol{\Omega})=\frac{c}{r}$$

(b) Consider a radiation field with angular fluence rate

$$\begin{aligned}\tilde{\varphi}(\boldsymbol{r},\boldsymbol{\Omega}) &= Ce^{-\lambda r\cdot}\ cos\theta \qquad 0\leq\theta\pi/2 \\ &= 0 \qquad \frac{\pi}{2}\leq\theta\leq\pi\end{aligned}$$

where C and λ are constants, and r is the cylinderised radial coordinate.

Find the net number of particles passing through the disc.

10. If one defines an average quantity <x> with respect to a function f(x) by

$$<x>_f=\frac{\int_0^\infty xf(x)dx}{\int_0^\infty f(x)dx}$$

find the relationship between the flux density average velocity and averages involving velocity with respect to the number density, n(v).

11. Given

$$\tilde{\varphi}(\Omega) = a + b\cos\theta$$

find the relationship between:

(i) a and the fluence rate.

(ii) b and the current density.

12. Given

$$\tilde{\varphi}(\bar{r},\bar{E}) = \varphi(z)\delta(E - E_0)\delta(1 - \bar{\Omega}\cdot\nabla z)$$

Describe the nature of the field. Assume the above describes a field of C K-X-rays (280 eV) incident on a cell culture. The cells are approximately cylinders of area A and thickness t, with axis aligned along z, and front surface at z=0. Given φ(z=0) = c and φ(z=t) = 0, find the total energy left in the cell using the energy flow vector.